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Minimal submanifolds in Riemannian spin manifolds with parallel spinor fields

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Abstract

The aim of this paper is to investigate certain minimal submanifolds in a $(2n + m)$-dimensional Riemannian spin manifold with a nonzero parallel spinor field. In particular, we give a characterization of compact totally geodesic hypersurfaces in such an ambient space. We also study minimal surfaces in a four-dimensional hyperkähler manifold from the viewpoint of spinors. As a result, we recover two results about minimal tori in a four-dimensional flat torus and minimal surfaces in a four-dimensional nonflat hyperkähler manifold. A Lichnerowicz type formula on a submanifold plays a key role in this paper.

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1. Introduction

In spin geometry, Lichnerowicz formula [6] is one of the fundamental tools and yields many important results in topology and Riemannian geometry (see [5]). Can we apply this formula to the study of differential geometry of submanifolds?

Recently Bär [1] introduced the “submanifold theory” of Dirac operators for the study of upper eigenvalue estimates for Dirac operators of closed hypersurfaces in real space forms. He compared the spin connection of the submanifold with the spin connection of the ambient space. As a result, he obtained a relation between the Dirac operator of the ambient space and that of the submanifold twisted with the spinor bundle of the normal bundle (see

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Lemma 2.1). This relation enables us to apply a Lichnerowicz type formula to the study of differential geometry of submanifolds.

In this paper, we study certain minimal submanifolds in a Riemannian spin manifold with parallel spinor fields. For example, a flat manifold and a Riemannian manifold which has one of the Ricci-flat holonomy groups $SU(n)$, $Sp(n)$, G_2 and $\mathrm{Spin}(7)$ are Riemannian spin manifolds equipped with a nonzero parallel spinor field. We shall consider the following problem:

Problem. Let Q be a $(2n+m)$-dimensional Riemannian spin manifold with a nonzero parallel spinor field. Let M be a $2n$-dimensional closed Riemannian spin manifold isometrically immersed in Q. Then classify all submanifolds M satisfying the condition that there exists a nonzero parallel spinor field $\psi \in \Gamma(\Sigma Q)$ such that

$$\nabla^{\Sigma M \otimes \Sigma N}(\psi|_M) = 0, \tag{$*$}$$

where $\psi|_M$ denotes the restriction of ψ to the submanifold M.
If M is totally geodesic in Q, then the condition $(*)$ will be satisfied (see Eq. (6)). Moreover, the condition $(*)$ implies that M is a minimal submanifold in Q (see Proposition 3.1). It seems worthwhile to characterize minimal submanifolds with the condition $(*)$ among all minimal submanifolds in Q.

This paper is organized as follows. In Section 2, we review necessary results from the "submanifold theory" of Dirac operators given by Bär. In Section 3, we derive a Lichnerowicz type formula for even-dimensional Riemannian spin submanifolds of arbitrary codimensions (Lemma 3.2). We apply this formula to study of the above problem and give a complete characterization of hypersurfaces with the condition $(*)$ (Theorem 3.5). In Section 4, we discuss minimal surfaces in a four-dimensional Riemannian spin manifold with a nonzero parallel spinor field. We give a complete answer to the above problem in the case where the ambient space is a four-dimensional flat torus or hyperkähler manifold (Theorems 4.4 and 4.5). Finally, we notice that very recently the spinorial techniques have been used for differential geometric study of surfaces (e.g. [3]) and hypersurfaces [4].

2. Bär's formulation of the submanifold theory of Dirac operators

In this section, we shall keep all the definitions and notations of Bär's paper [1]. We first review Clifford algebras and their representations (cf. [8]). Let E be an n-dimensional oriented Euclidean vector space. We denote by $\mathbf{C}l(E)$ the complex Clifford algebra of E, i.e. the complexification of the Clifford algebra $Cl(E)$ generated by all elements of E. The dimension of $\mathbf{C}l(E)$ as a complex vector space is 2^n. Fix an oriented orthonormal basis $e_1, \ldots, e_n$ for E. The complex volume element $\omega_{\mathbf{C}}$ is defined by

$$\omega_{\mathbf{C}} = \sqrt{-1}^{[(n+1)/2]} e_1 \cdots e_n.$$

We can easily check that $\omega_{\mathbf{C}}^2 = 1$ and that $\omega_{\mathbf{C}}$ is independent of the choice of oriented orthonormal basis.

If n is even, then $\mathbf{C}l(E)$ is isomorphic to the matrix algebra $M(2^{n/2}, \mathbf{C})$ and so has only one irreducible module, which is denoted by ΣE. In other words, we have a unique irreducible complex representation of $Cl(E)$ with representation space ΣE. We denote the Clifford multiplication by $\gamma_E : \mathbf{C}l(E) \to \mathrm{End}(\Sigma E)$. When restricted to the even subalgebra $\mathbf{C}l_0(E) := Cl_0(E) \otimes \mathbf{C}$, ΣE decomposes into a direct sum $\Sigma^+ E \oplus \Sigma^- E$ as $\mathbf{C}l_0(E)$-modules. Here $\Sigma^\pm E$ denote the eigenspace with eigenvalue ± 1 for the action of $\gamma_E(\omega_{\mathbf{C}})$, respectively.

If n is odd, then $\mathbf{C}l(E)$ is isomorphic to $M(2^{(n-1)/2}, \mathbf{C}) \oplus M(2^{(n-1)/2}, \mathbf{C})$ as algebra and hence has only two irreducible modules, $\Sigma^0 E$ and $\Sigma^1 E$. We denote the Clifford multiplication by $\gamma_{E,j} : \mathbf{C}l(E) \to \mathrm{End}(\Sigma^j E)$, $j = 0, 1$. The modules $\Sigma^0 E$ and $\Sigma^1 E$ can be distinguished by the action of $\omega_{\mathbf{C}}$. On $\Sigma^j E$ it acts as $(-1)^j$, $j = 0, 1$.

Next, we explain a representation of the Clifford algebra of a direct sum of two Euclidean vector spaces constructed from each factor. Let E and F be oriented Euclidean vector spaces of dimension n and m, respectively. We review this representation in the case where n is even.

Case 1 (n and m are even). We put

$$\Sigma := \Sigma E \otimes \Sigma F,$$

and define the map

$$\gamma : E \oplus F \to \mathrm{End}(\Sigma)$$

by

$$\gamma(X)(\sigma \otimes \tau) := (\gamma_E(X)\sigma) \otimes \tau, \qquad \gamma(Y)(\sigma \otimes \tau) := (-1)^{\deg \sigma} \sigma \otimes \gamma_F(Y)\tau,$$

where $X \in E$, $Y \in F$, $\tau \in \Sigma F$ and $\sigma \in \Sigma^+ E$ or $\sigma \in \Sigma^- E$. Here $\deg \sigma$ is defined by the equation $\gamma_E(\omega_{\mathbf{C}})\sigma = (-1)^{\deg \sigma}\sigma$. Then we can easily check that

$$\gamma(X + Y)\gamma(X + Y)(\sigma \otimes \tau) = -|X + Y|^2 \sigma \otimes \tau.$$

Hence γ extends to a homomorphism $\gamma : \mathbf{C}l(E \oplus F) \to \mathrm{End}(\Sigma)$. That is to say, (Σ, γ) is a nontrivial $\mathbf{C}l(E \oplus F)$-module of complex dimension $2^{n/2} \cdot 2^{m/2} = 2^{(n+m)/2}$. Since $\dim_{\mathbf{C}}(\Sigma(E \oplus F)) = 2^{(n+m)/2}$, we have

$$(\Sigma, \gamma) \simeq (\Sigma(E \oplus F), \gamma_{E \oplus F}).$$

The decomposition into positive and negative part is given by

$$\Sigma^+(E \oplus F) = (\Sigma^+ E \otimes \Sigma^+ F) \oplus (\Sigma^- E \otimes \Sigma^- F),$$
$$\Sigma^-(E \oplus F) = (\Sigma^+ E \otimes \Sigma^- F) \oplus (\Sigma^- E \otimes \Sigma^+ F).$$

Case 2 (n is even and m is odd). We put

$$\Sigma^j := \Sigma E \otimes \Sigma^j F$$

for $j = 0, 1$ and define the maps

$$\gamma_j : E \oplus F \to \mathrm{End}(\Sigma^j)$$

by the same way in Case 1. Then (Σ^0, γ_0) and (Σ^1, γ_1) become nontrivial $\mathbf{C}l(E \oplus F)$-modules of complex dimension $2^{n/2} \cdot 2^{(m-1)/2} = 2^{(n+m-1)/2}$, and $\dim_{\mathbf{C}}(\Sigma^j(E \oplus F)) =$

$2^{((n+m)-1)/2}$, $j = 0, 1$. Since we can prove that the complex volume element $\omega_{\mathbf{C}}$ of $\mathbf{C}l(E \oplus F)$ acts on Σ^j as $(-1)^j$, we have

$$(\Sigma^j, \gamma_j) \simeq (\Sigma^j(E \oplus F), \gamma_{E\oplus F,j}), \quad j = 0, 1.$$

Under the above algebraic preliminaries, we review the "submanifold theory" of Dirac operators by Bär. Let Q be an $(n+m)$-dimensional oriented Riemannian manifold and let $M \hookrightarrow Q$ be an n-dimensional oriented immersed submanifold. Throughout this paper we assume, unless stated otherwise, that M carries the induced Riemannian metric. We suppose that both Riemannian manifolds M and Q are equipped with spin structures. These induce a unique spin structure on the normal bundle N of M in Q by Milnor's lemma (see [5, p. 85]).

We denote the Levi-Civita connections of M and Q by ∇^M and ∇^Q, respectively. Let ∇^N be the normal connection on N and let $I\!I$ be the second fundamental form of M in Q. For $p \in M$ and $X \in T_pM$, we have

$$\nabla^Q_X = \begin{pmatrix} \nabla^M_X & -I\!I(X,\cdot)^* \\ I\!I(X,\cdot) & \nabla^N_X \end{pmatrix} \tag{1}$$

with respect to the decomposition $T_pQ = T_pM \oplus N_p$.

Let $X_1, \ldots, X_n$ be a positively oriented local orthonormal frame of TM near p. Let $Y_1, \ldots, Y_m$ be a positively oriented local orthonormal frame of N near p. Then $h := (X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ is a local section of the frame bundle of Q restricted to M and Eq. (1) is represented in matrix form as

$$\nabla^Q_X - (\nabla^M_X \oplus \nabla^N_X) = \begin{pmatrix} 0 & (-\langle I\!I(X,X_i), Y_j\rangle)_{ij} \\ (\langle I\!I(X,X_i), Y_j\rangle)_{ji} & 0 \end{pmatrix}. \tag{2}$$

Let ω^M, ω^N and ω^Q be the connection 1-forms of ∇^M, ∇^N and ∇^Q lifted to $\mathfrak{spin}(n)$, $\mathfrak{spin}(m)$ and $\mathfrak{spin}(n+m)$, respectively. We denote the standard double covering map by $\Theta : \mathrm{Spin}(n+m) \to SO(n+m)$. Then Eq. (2) yields

$$\begin{aligned}
&\Theta_*(\omega^Q((\mathrm{d}h)_pX) - (\omega^M \oplus \omega^N)((\mathrm{d}h)_pX)) \\
&\quad = \begin{pmatrix} 0 & (-\langle I\!I(X,X_i), Y_j\rangle)_{ij} \\ (\langle I\!I(X,X_i), Y_j\rangle)_{ji} & 0 \end{pmatrix} \\
&\quad = \sum_{i=1}^n \sum_{j=1}^m \langle I\!I(X,X_i), Y_j\rangle \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & -1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \begin{matrix} (i \\ \\ (n+j \end{matrix} \\
&\quad = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^m \langle I\!I(X,X_i), Y_j\rangle \Theta_*(e_i \cdot f_j) = \Theta_*\left(\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^m \langle I\!I(X,X_i), Y_j\rangle e_i \cdot f_j\right),
\end{aligned} \tag{3}$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbf{R}^n$ and $f_1, \dots, f_m$ is the standard basis of $\mathbf{R}^m$. Therefore, Eq. (3) yields

$$\omega^Q((\mathrm{d}h)_p X) - (\omega^M \oplus \omega^N)((\mathrm{d}h)_p X) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\langle II(X, X_i), Y_j\rangle(p) e_i \cdot f_j. \tag{4}$$

We denote the complex spinor bundles associated to spin structures of TQ, TM and N by ΣQ, ΣM and ΣN, respectively. Since $\mathrm{Spin}(n)$ is compact, these bundles carry Hermitian inner products unique up to isomorphism. For example, ΣM is induced by the action of $\mathbf{C}l(E)$ on ΣE, so we can choose the metric to be invariant under the action of $\mathrm{Pin}(n)$ (e.g. [8, p. 24]). From the previous algebraic arguments, we have $\Sigma Q|_M = \Sigma M \otimes \Sigma N$ if n is even. Let $\nabla^{\Sigma Q}$, $\nabla^{\Sigma M}$ and $\nabla^{\Sigma N}$ be the spin connections on ΣQ, ΣM and ΣN, respectively. The product connection on $\Sigma M \otimes \Sigma N$ is defined by

$$\nabla^{\Sigma M \otimes \Sigma N} := \nabla^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\Sigma N}.$$

Eq. (4) yields

$$\begin{aligned}\nabla_X^{\Sigma Q} - \nabla_X^{\Sigma M \otimes \Sigma N} &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\langle II(X, X_i), Y_j\rangle \gamma_Q(X_i \cdot Y_j) \\ &= \frac{1}{2}\sum_{i=1}^{n}\gamma_Q\left(X_i \cdot \sum_{j=1}^{m}\langle II(X, X_i), Y_j\rangle Y_j\right),\end{aligned} \tag{5}$$

$$\nabla_X^{\Sigma Q} - \nabla_X^{\Sigma M \otimes \Sigma N} = \frac{1}{2}\sum_{i=1}^{n}\gamma_Q(X_i \cdot II(X, X_i)). \tag{6}$$

Here γ_Q means the Clifford multiplication on $\Sigma Q|_M$. Similarly, we denote the Clifford multiplication on ΣM and on ΣN by γ_M and γ_N, respectively.

We explain two Dirac type operators. We denote by $D_M^{\Sigma N}$ the Dirac operator on M twisted with the complex spinor bundle ΣN, that is

$$D_M^{\Sigma N} := \sum_{j=1}^{n}\gamma_Q(X_j)\nabla_{X_j}^{\Sigma M \otimes \Sigma N} = \sum_{j=1}^{n}(\gamma_M(X_j) \otimes \mathrm{Id})\nabla_{X_j}^{\Sigma M \otimes \Sigma N}.$$

This is a formally self-adjoint operator. And another operator $\hat{D}$ is defined as

$$\hat{D} := \sum_{j=1}^{n}\gamma_Q(X_j)\nabla_{X_j}^{\Sigma Q}.$$

We can easily check that the above definitions are independent of the choice of oriented orthonormal frame $X_1, \dots, X_n$. Both operators act on sections of $\Sigma Q|_M$. Since $\Sigma Q|_M = \Sigma M \otimes \Sigma N$, we can consider that they also act on sections of $\Sigma M \otimes \Sigma N$. Let $H := (1/n)\sum_{i=1}^{n} II(X_i, X_i)$ be the mean curvature vector field of M in Q. The following formula plays an important role in this paper.

Lemma 2.1 (Bär [1]). *If n is even,*

$$D_M^{\Sigma N} = \hat{D} + \tfrac{1}{2} n \gamma_Q(H).$$

Proof. By Eq. (6),

$$\begin{aligned}\hat{D} - D_M^{\Sigma N} &= \sum_{j=1}^{n} \gamma_Q(X_j)(\nabla_{X_j}^{\Sigma Q} - \nabla_{X_j}^{\Sigma M \otimes \Sigma N}) = \frac{1}{2} \sum_{j=1}^{n} \gamma_Q(X_j) \sum_{i=1}^{n} \gamma_Q(X_i \cdot II(X_j, X_i)) \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \gamma_Q(X_j \cdot X_i) \gamma_Q(II(X_j, X_i)).\end{aligned}$$

Since $\gamma_Q(X_j \cdot X_i) + \gamma_Q(X_i \cdot X_j) = 0$ for $i \neq j$ and $II(X_j, X_i) = II(X_i, X_j)$, we have

$$\begin{aligned}\hat{D} - D_M^{\Sigma N} &= \frac{1}{2} \sum_{i=1}^{n} \gamma_Q(X_i \cdot X_i) \gamma_Q(II(X_i, X_i)) \\ &= -\frac{1}{2} \gamma_Q \left(\sum_{i=1}^{n} II(X_i, X_i) \right) = -\frac{n}{2} \gamma_Q(H).\end{aligned}$$

□

3. Even-dimensional submanifolds in Riemannian spin manifolds

The purpose of this paper is to study minimal submanifolds with the condition (∗) of a Riemannian spin manifold with a nonzero parallel spinor field. We denote by $\Gamma(V)$ the set of the sections of a real or complex vector bundle V. Recall that a spinor field $\psi \in \Gamma(\Sigma Q)$ is said to be *parallel* if $\nabla_X^{\Sigma Q}(\psi) = 0$ for all $X \in \Gamma(TQ)$. We can easily check that the set of parallel spinor fields on Q forms a finite-dimensional vector subspace of $\Gamma(\Sigma Q)$ and that each parallel spinor field has constant length. First of all, we show that the condition (∗) yields the minimality of a submanifold M in Q.

Proposition 3.1. *Let Q^{2n+m} be a $(2n + m)$-dimensional Riemannian spin manifold with a nonzero parallel spinor field. Let M^{2n} be a $2n$-dimensional Riemannian spin manifold isometrically immersed in Q^{2n+m}. Let the normal bundle N of M^{2n} in Q^{2n+m} carry the induced spin structure. If M^{2n} satisfies the condition (∗), then M^{2n} is a minimal submanifold in Q^{2n+m}.*

Proof. By the assumption, there exists a nonzero parallel spinor field $\psi \in \Gamma(\Sigma Q)$ such that $\nabla^{\Sigma M \otimes \Sigma N}(\psi|_M) = 0$. Of course, $\psi|_M \in \Gamma(\Sigma Q|_M)$ has nonzero constant length. By Lemma 2.1, we have

$$D_M^{\Sigma N}(\psi|_M) = \sum_{i=1}^{2n} \gamma_Q(X_i) \nabla_{X_i}^{\Sigma Q}(\psi|_M) + n\gamma_Q(H)\psi|_M = n\gamma_Q(H)\psi|_M.$$

Moreover

$$D_M^{\Sigma N}(\psi|_M)=\sum_{i=1}^{2n}\gamma_Q(X_i)\nabla_{X_i}^{\Sigma M\otimes\Sigma N}(\psi|_M)=0.$$

So we have $\gamma_Q(H)\psi|_M=0$. Taking pointwise Hermitian inner product,

$$|H|^2\langle\psi|_M,\psi|_M\rangle=0.$$

Since $\langle\psi|_M,\psi|_M\rangle\neq 0$, we obtain $H=0$. □

Next we show the following Lichnerowicz type formula. This is a fundamental formula in our theory. For simplicity $\nabla^{\Sigma M\otimes\Sigma N}$ is denoted by ∇ at times.

Lemma 3.2 (Lichnerowicz type formula). *Let Q^{2n+m} be a $(2n+m)$-dimensional Riemannian spin manifold. Let M^{2n} be a $2n$-dimensional Riemannian spin manifold isometrically immersed in Q^{2n+m}. Let the normal bundle N of M^{2n} in Q^{2n+m} carry the induced spin structure. We denote the scalar curvature of M^{2n} by κ and the curvature tensor field of the normal connection ∇^N by $R^\perp$. Then we have*

$$(D_M^{\Sigma N})^2=\nabla^*\nabla+\frac{1}{4}\kappa+\frac{1}{2}\gamma_Q\left(\sum_{i<j}^{2n}\sum_{k<l}^{m}\langle R^\perp_{X_i,X_j}(Y_k),Y_l\rangle X_i\cdot X_j\cdot Y_k\cdot Y_l\right).$$

Proof. By the formula (8.23) in [5, p. 164], it suffices to calculate $\mathcal{R}^{\Sigma N}:\Gamma(\Sigma M\otimes\Sigma N)\to\Gamma(\Sigma M\otimes\Sigma N)$ defined by the formula

$$\begin{aligned}\mathcal{R}^{\Sigma N}(\sigma\otimes\tau)&:=\frac{1}{2}\sum_{i,j=1}^{2n}\gamma_M(X_i\cdot X_j)\sigma\otimes(R^{\Sigma N}_{X_i,X_j}\tau)\\&=\sum_{i<j}^{2n}\gamma_M(X_i\cdot X_j)\sigma\otimes(R^{\Sigma N}_{X_i,X_j}\tau)\end{aligned}\tag{7}$$

on $\sigma\otimes\tau$, where $\sigma\in\Gamma(\Sigma^+M)$ or $\sigma\in\Gamma(\Sigma^-M)$, and $\tau\in\Gamma(\Sigma N)$. Here $R^{\Sigma N}$ means the curvature tensor field of the connection $\nabla^{\Sigma N}$. And by the formula (4.37) in [5, p. 110], we have

$$R^{\Sigma N}_{X_i,X_j}(\tau)=\frac{1}{2}\sum_{k<l}^{m}\langle R^\perp_{X_i,X_j}(Y_k),Y_l\rangle\gamma_N(Y_k\cdot Y_l)\tau.\tag{8}$$

Plugging Eq. (8) in (7), we obtain

$$\begin{aligned}\mathcal{R}^{\Sigma N}(\sigma\otimes\tau) &= \frac{1}{2}\sum_{i<j}^{2n}\sum_{k<l}^{m}\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle\gamma_M(X_i\cdot X_j)\sigma\otimes\gamma_N(Y_k\cdot Y_l)\tau \\ &= \frac{1}{2}\sum_{i<j}^{2n}\sum_{k<l}^{m}\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle\gamma_Q(X_i\cdot X_j)(\sigma\otimes\gamma_N(Y_k\cdot Y_l)\tau) \\ &= \frac{1}{2}\sum_{i<j}^{2n}\sum_{k<l}^{m}\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle\gamma_Q(X_i\cdot X_j\cdot Y_k\cdot Y_l)(\sigma\otimes\tau) \\ &= \frac{1}{2}\gamma_Q\left(\sum_{i<j}^{2n}\sum_{k<l}^{m}\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle X_i\cdot X_j\cdot Y_k\cdot Y_l\right)(\sigma\otimes\tau). \qquad\square\end{aligned}$$

Corollary 3.3. *Let Q^{2n+m} be a $(2n+m)$-dimensional Riemannian spin manifold with a nonzero parallel spinor field. Let M^{2n} be a $2n$-dimensional closed Riemannian spin manifold isometrically immersed in Q^{2n+m}. Let the normal bundle N carry the induced spin structure. We denote the scalar curvature of M^{2n} by κ and the mean curvature vector field of M^{2n} in Q^{2n+m} by H. Then we have*

$$\int_M\left(n^2|H|^2-\frac{\kappa}{4}+\frac{1}{2}\sum_{i<j}^{2n}\sum_{k<l}^{m}|\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle|\right)\,\mathrm{dvol}\geq 0.$$

Proof. Let $\psi\in\Gamma(\Sigma Q)$ be a parallel spinor field on Q such that $|\psi|\equiv 1$. Then $\psi|_M\in\Gamma(\Sigma Q|_M)$ also has constant length 1. By Lemma 2.1,

$$D_M^{\Sigma N}(\psi|_M)=\sum_{i=1}^{2n}\gamma_Q(X_i)\nabla^{\Sigma Q}_{X_i}(\psi|_M)+n\gamma_Q(H)\psi|_M=n\gamma_Q(H)\psi|_M.$$

Hence we obtain

$$\begin{aligned}\int_M\langle D_M^{\Sigma N}(\psi|_M), D_M^{\Sigma N}(\psi|_M)\rangle\,\mathrm{dvol} &= n^2\int_M\langle\gamma_Q(H)\psi|_M,\gamma_Q(H)\psi|_M\rangle\,\mathrm{dvol} \\ &= n^2\int_M|H|^2\langle\psi|_M,\psi|_M\rangle\,\mathrm{dvol}=n^2\int_M|H|^2\,\mathrm{dvol}.\end{aligned}$$

On the other hand, by Lemma 3.2,

$$\begin{aligned}(D_M^{\Sigma N})^2\psi|_M &= \nabla^*\nabla(\psi|_M)+\frac{1}{4}\kappa\psi|_M \\ &\quad+\frac{1}{2}\gamma_Q\left(\sum_{i<j}^{2n}\sum_{k<l}^{m}\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle X_i\cdot X_j\cdot Y_k\cdot Y_l\right)\psi|_M.\end{aligned}$$

Taking pointwise Hermitian inner product with $\psi|_M$ and integrating over M,

$$\int_M \langle (D_M^{\Sigma N})^2 \psi|_M, \psi|_M\rangle \,\mathrm{dvol}$$
$$= \int_M \langle \nabla^*\nabla(\psi|_M), \psi|_M\rangle \,\mathrm{dvol} + \frac{1}{4}\int_M \kappa\langle \psi|_M, \psi|_M\rangle \,\mathrm{dvol}$$
$$+ \frac{1}{2}\int_M \left\langle \gamma_Q\left(\sum_{i<j}^{2n}\sum_{k<l}^{m} \langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle X_i\cdot X_j\cdot Y_k\cdot Y_l\right)\psi|_M, \psi|_M\right\rangle \mathrm{dvol}.$$

Since $D_M^{\Sigma N}$ is formally self-adjoint, we have

$$n^2\int_M |H|^2 \,\mathrm{dvol} = \int_M |\nabla(\psi|_M)|^2 \,\mathrm{dvol} + \frac{1}{4}\int_M \kappa \,\mathrm{dvol}$$
$$+ \frac{1}{2}\int_M \left\langle \gamma_Q\left(\sum_{i<j}^{2n}\sum_{k<l}^{m} \langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle X_i\cdot X_j\cdot Y_k\cdot Y_l\right)\psi|_M, \psi|_M\right\rangle \mathrm{dvol}, \tag{9}$$

where

$$\left|\left\langle \gamma_Q\left(\sum_{i<j}^{2n}\sum_{k<l}^{m} \langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle X_i\cdot X_j\cdot Y_k\cdot Y_l\right)\psi|_M, \psi|_M\right\rangle\right|$$
$$\leq \sum_{i<j}^{2n}\sum_{k<l}^{m} |\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle||\langle \gamma_Q(X_i\cdot X_j\cdot Y_k\cdot Y_l)\psi|_M, \psi|_M\rangle|$$
$$\leq \sum_{i<j}^{2n}\sum_{k<l}^{m} |\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle||\gamma_Q(X_i\cdot X_j\cdot Y_k\cdot Y_l)\psi|_M||\psi|_M|$$
$$= \sum_{i<j}^{2n}\sum_{k<l}^{m} |\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle||\psi|_M|^2 = \sum_{i<j}^{2n}\sum_{k<l}^{m} |\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle|.$$

Hence

$$n^2\int_M |H|^2 \,\mathrm{dvol} \geq \int_M |\nabla(\psi|_M)|^2 \,\mathrm{dvol} + \frac{1}{4}\int_M \kappa \,\mathrm{dvol}$$
$$-\frac{1}{2}\int_M \sum_{i<j}^{2n}\sum_{k<l}^{m} |\langle R^{\perp}_{X_i,X_j}(Y_k), Y_l\rangle| \,\mathrm{dvol}.$$

Therefore, we obtain the conclusion. □

Proposition 3.4. *Let Q^{2n+m} be a $(2n+m)$-dimensional Riemannian spin manifold with a nonzero parallel spinor field. Let M^{2n} be a $2n$-dimensional closed Riemannian spin manifold isometrically immersed in Q^{2n+m}. Let the normal bundle N carry the induced spin structure. If M^{2n} is totally geodesic in Q^{2n+m} and the normal connection $\nabla^{\perp}$ is flat, then the Ricci tensor field of M^{2n} is zero.*

Proof. Let $\psi \in \Gamma(\Sigma Q)$ be a parallel spinor field on Q such that $|\psi| \equiv 1$. Then $\psi|_M \in \Gamma(\Sigma Q|_M)$ and $|\psi|_M| \equiv 1$. By Eq. (6),

$$\nabla_X^{\Sigma M \otimes \Sigma N}(\psi|_M) = 0$$

for any $X \in \Gamma(TM)$. Hence the curvature tensor field $R^{\Sigma M \otimes \Sigma N}$ of $\nabla^{\Sigma M \otimes \Sigma N}$ satisfies that

$$R^{\Sigma M \otimes \Sigma N}(\psi|_M) = 0.$$

By definition, $R^{\Sigma M \otimes \Sigma N} = R^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes R^{\Sigma N}$.

Fix $p \in M$. Let U be an open neighborhood of the point p in M. Let $\{\tau_j\}$ be an oriented orthonormal frame of ΣN on U. Then we have

$$\psi|_M = \sum_j \sigma_j \otimes \tau_j,$$

where each σ_j stands for a section of ΣM on U, and

$$0 = R^{\Sigma M \otimes \Sigma N}(\psi|_M) = \sum_j (R^{\Sigma M}\sigma_j) \otimes \tau_j + \sum_j \sigma_j \otimes R^{\Sigma N}\tau_j.$$

Since $R^{\perp} = 0$, Eq. (8) implies that $R^{\Sigma N} = 0$, so

$$\sum_j (R^{\Sigma M}\sigma_j) \otimes \tau_j = 0.$$

Hence we have $R^{\Sigma M}\sigma_j = 0$ for all j. We may assume that one of the σ_j's never vanishes on U. By the formula (1.13) in [2, p. 16], we have

$$0 = \sum_{i=1}^{2n} \gamma_M(X_i) R^{\Sigma M}_{X, X_i}(\sigma_j) = -\frac{1}{2}\gamma_M(\mathrm{Ric}(X))\sigma_j,$$

where $\mathrm{Ric}(X)$ is defined as $\mathrm{Ric}(X) := \sum_{i=1}^{2n} \mathrm{Ric}(X, X_i)X_i$. Since $\sigma_j \neq 0$ on U, we obtain $\mathrm{Ric}(X) = 0$ for any $X \in \Gamma(TM)$ on U. □

If M is an oriented hypersurface immersed in a Riemannian spin manifold Q, then the normal bundle N is an oriented real line bundle, hence trivial. Therefore, the complex line bundle ΣN is also trivial. In particular, M carries the induced spin structure from the ambient space Q.

Theorem 3.5. *Let Q^{2n+1} be a $(2n+1)$-dimensional Riemannian spin manifold with a nonzero parallel spinor field. Let M^{2n} be a $2n$-dimensional closed oriented Riemannian manifold isometrically immersed in Q^{2n+1}. Let M^{2n} carry the induced spin structure. Then the following conditions are equivalent*:

1. *M^{2n} is a minimal hypersurface in Q^{2n+1} and the scalar curvature of M^{2n} is identically zero.*
2. *M^{2n} is a minimal hypersurface in Q^{2n+1} and the Ricci tensor field of M^{2n} is zero.*

3. *M^{2n} is totally geodesic in Q^{2n+1}.*
4. *$\nabla^{\Sigma M\otimes\Sigma N}(\psi|_M)=0$ for any parallel spinor field $\psi\in\Gamma(\Sigma Q)$.*
5. *M^{2n} satisfies the condition $(*)$.*

Proof. (1)⇒(3): Let $\psi\in\Gamma(\Sigma Q)$ be a parallel spinor field on Q such that $|\psi|\equiv 1$. Then $\psi|_M\in\Gamma(\Sigma Q|_M)$ and $|\psi|_M|\equiv 1$. Suppose that $\kappa=0$. Then Eq. (9) yields $\nabla^{\Sigma M\otimes\Sigma N}(\psi|_M)=0$. By Eq. (5), we obtain

$$0=\nabla_X^{\Sigma Q}(\psi|_M)-\nabla_X^{\Sigma M\otimes\Sigma N}(\psi|_M)=\frac{1}{2}\sum_{i=1}^{2n}\langle II(X,X_i),Y_1\rangle\gamma_Q(X_i\cdot Y_1)\psi|_M$$

for any $X\in\Gamma(TM)$. Here Y_1 is a local unit normal vector field on M. Fix $p\in M$ and put $\varphi:=\gamma_Q(Y_1)\psi|_M$. Then $\langle\varphi,\varphi\rangle=1$ and

$$\sum_{i=1}^{2n}\langle II(X,X_i),Y_1\rangle\gamma_Q(X_i)\varphi=0.$$

Taking Hermitian inner product with $\gamma_Q(X_j)\varphi$, then

$$\sum_{i\neq j}\langle II(X,X_i),Y_1\rangle\langle\gamma_Q(X_i)\varphi,\gamma_Q(X_j)\varphi\rangle+\langle II(X,X_j),Y_1\rangle=0$$

for $j=1,\ldots,2n$. The real parts of these equations are

$$\langle II(X,X_j),Y_1\rangle(p)=0,\qquad j=1,\ldots,2n.$$

Hence M is totally geodesic in Q.

(3)⇒(4): Assume that $II=0$. Then Eq. (6) yields

$$\nabla^{\Sigma M\otimes\Sigma N}(\psi|_M)=0$$

for any parallel spinor field $\psi\in\Gamma(\Sigma Q)$.

(4)⇒(5): Trivial.

(5)⇒(1): By the assumption, there exists a nonzero parallel spinor field $\psi\in\Gamma(\Sigma Q)$ such that

$$\nabla^{\Sigma M\otimes\Sigma N}(\psi|_M)=0.$$

By Proposition 3.1, it suffices to show the latter part of the statement of (1). By Lemma 3.2, we have

$$(D_M^{\Sigma N})^2\psi|_M=\tfrac{1}{4}\kappa\psi|_M.$$

The condition (5) also implies that $D_M^{\Sigma N}(\psi|_M)=0$. Hence $\kappa\psi|_M=0$. Since $\psi|_M$ is nonzero, we conclude that $\kappa=0$.

We have just proved that (1)⇒(3)⇒(4)⇒(5)⇒(1). Since (2) implies (1) by their definitions, it suffices to show that (3)⇒(2). But this is a direct consequence of Proposition 3.4. □

Nontrivial examples of hypersurfaces in Theorem 3.5 appear, for instance, in a nonflat five-dimensional Riemannian manifold constructed by Friedrich and Kath [2, Chapter 6]. They described all nonflat compact five-dimensional Riemannian manifolds with a nonzero parallel spinor field. Such a manifold Q^5 is a total space of a fiber bundle over S^1. Each fiber is a totally geodesic $K3$ surface with a Ricci-flat Kähler metric.

We also remark that the equivalence of (1) and (3) in Theorem 3.5 is obtained by using the equation of Gauss under the assumption that the ambient space Q^{2n+1} is only Ricci-flat.

4. Surfaces in four-dimensional Riemannian spin manifolds

Lemma 4.1 (Lichnerowicz type formula). *Let Q^4 be a four-dimensional Riemannian spin manifold. Let M^2 be a two-dimensional Riemannian spin manifold isometrically immersed in Q^4. We denote the Gaussian curvature of M^2 by K and the curvature of the normal bundle N by K_N. Then we have the following formulae*:

1. *On $\Gamma(\Sigma^+ Q|_M) = \Gamma(\Sigma^+ M \otimes \Sigma^+ N) \oplus \Gamma(\Sigma^- M \otimes \Sigma^- N)$, it holds*
$$(D_M^{\Sigma N})^2 = \nabla^*\nabla + \tfrac{1}{2}K + \tfrac{1}{2}K_N.$$
2. *On $\Gamma(\Sigma^- Q|_M) = \Gamma(\Sigma^+ M \otimes \Sigma^- N) \oplus \Gamma(\Sigma^- M \otimes \Sigma^+ N)$, it holds*
$$(D_M^{\Sigma N})^2 = \nabla^*\nabla + \tfrac{1}{2}K - \tfrac{1}{2}K_N.$$

Proof. We first remark that the scalar curvature is equal to twice the Gaussian curvature K on a two-dimensional Riemannian manifold and that the normal curvature K_N is defined as $K_N := \langle R^{\perp}_{X_1,X_2}(Y_2), Y_1\rangle$. Let $\sigma^+ \in \Gamma(\Sigma^+ M)$ and $\tau^+ \in \Gamma(\Sigma^+ N)$. By the same way as the proof of Lemma 3.2, we have

$$\begin{aligned}\mathcal{R}^{\Sigma N}(\sigma^+ \otimes \tau^+) &= \gamma_M(X_1 \cdot X_2)\sigma^+ \otimes (R^{\Sigma N}_{X_1,X_2}\tau^+)\\ &= \gamma_M(X_1 \cdot X_2)\sigma^+ \otimes \tfrac{1}{2}\langle R^{\perp}_{X_1,X_2}(Y_1), Y_2\rangle \gamma_N(Y_1 \cdot Y_2)\tau^+\\ &= \tfrac{1}{2}K_N \gamma_M(\sqrt{-1}X_1 \cdot X_2)\sigma^+ \otimes \gamma_N(\sqrt{-1}Y_1 \cdot Y_2)\tau^+\\ &= \tfrac{1}{2}K_N(\sigma^+ \otimes \tau^+).\end{aligned}$$

For $\sigma^- \in \Gamma(\Sigma^- M)$ and $\tau^- \in \Gamma(\Sigma^- N)$, we also have

$$\begin{aligned}\mathcal{R}^{\Sigma N}(\sigma^- \otimes \tau^-) &= \tfrac{1}{2}K_N \gamma_M(\sqrt{-1}X_1 \cdot X_2)\sigma^- \otimes \gamma_N(\sqrt{-1}Y_1 \cdot Y_2)\tau^-\\ &= \tfrac{1}{2}K_N(-\sigma^-) \otimes (-\tau^-) = \tfrac{1}{2}K_N(\sigma^- \otimes \tau^-).\end{aligned}$$

The above two formulae yield (1). Similarly, we can prove (2). □

Proposition 4.2. *Let Q^4 be a four-dimensional Riemannian spin manifold. Let M^2 be a closed oriented surface immersed in Q^4. Let M^2 carry the induced Riemannian metric.*

1. *If Q^4 admits a nonzero parallel spinor field $\psi^+ \in \Gamma(\Sigma^+ Q)$, then*

$$\int_M |H|^2 \,\mathrm{dvol} = \int_M |\nabla^{\Sigma M \otimes \Sigma N}(\psi^+|_M)|^2 \,\mathrm{dvol} + \pi(\chi(M) + \chi(N)).$$

2. *If Q^4 admits a nonzero parallel spinor field $\psi^- \in \Gamma(\Sigma^- Q)$, then*

$$\int_M |H|^2 \,\mathrm{dvol} = \int_M |\nabla^{\Sigma M \otimes \Sigma N}(\psi^-|_M)|^2 \,\mathrm{dvol} + \pi(\chi(M) - \chi(N)).$$

Here $\chi(M)$ and $\chi(N)$ denote the Euler numbers of M and its normal bundle, respectively, and we normalize $\psi^\pm$ such that $|\psi^\pm| \equiv 1$.

Proof. We first prove (1). Remark that the second Stiefel–Whitney class $w_2(M)$ of M is just the mod 2 reduction of the Euler class (see [5, p. 82]). Hence $w_2(M)$ is zero and there exists a spin structure on M. If we fix a spin structure on M, then N carries the induced spin structure. Let $\psi^+ \in \Gamma(\Sigma^+ Q)$ be a parallel spinor field on Q such that $|\psi^+| \equiv 1$. Then $\psi^+|_M \in \Gamma(\Sigma^+ Q|_M)$ and $|\psi^+|_M| \equiv 1$. By a similar calculation in Corollary 3.3, Lemma 4.1(1) yields

$$\int_M |H|^2 \,\mathrm{dvol} = \int_M |\nabla^{\Sigma M \otimes \Sigma N}(\psi^+|_M)|^2 \,\mathrm{dvol} + \frac{1}{2}\int_M K \,\mathrm{dvol} + \frac{1}{2}\int_M K_N \,\mathrm{dvol}.$$

Using the Gauss–Bonnet formula and $\int_M K_N \,\mathrm{dvol} = 2\pi\chi(N)$ (see [[7], Proposition 3.3]), we complete the proof of (1). Similarly, Lemma 4.1(2) yields the claim of (2). □

Since a four-dimensional flat torus has not only a nonzero positive parallel spinor field but also negative one, Proposition 4.2 yields the following lemma.

Lemma 4.3. *Let Q^4 be a four-dimensional flat torus. Let M^2 be a closed oriented surface immersed in Q^4. Let M^2 carry the induced Riemannian metric. If M^2 is a minimal surface in Q^4, then*

$$\chi(M) + |\chi(N)| \le 0.$$

Under the foregoing preliminaries, we shall consider the problem which we mentioned at the beginning of this paper.

Theorem 4.4. *Let Q^4 be a four-dimensional flat torus. Let M^2 be a closed oriented surface of genus one immersed in Q^4. Let M^2 carry the induced Riemannian metric. Then the following conditions are equivalent*:

1. *M^2 is a minimal surface in Q^4.*
2. *M^2 satisfies the condition* $(*)$.
3. *M^2 is totally geodesic in Q^4.*
4. $\nabla^{\Sigma M \otimes \Sigma N}(\psi^+|_M) = 0$ *for any parallel spinor field* $\psi^+ \in \Gamma(\Sigma^+ Q)$ *and* $\nabla^{\Sigma M \otimes \Sigma N}(\psi^-|_M) = 0$ *for any parallel spinor field* $\psi^- \in \Gamma(\Sigma^- Q)$.

Proof. (3)⇒(2): Trivial.

(2)⇒(1): Trivial by Proposition 3.1.

(1)⇒(4): Since $\chi(M) = 0$, we have $\chi(N) = 0$ by Lemma 4.3. Moreover, $H = 0$ and Proposition 4.2 imply that

$$\nabla^{\Sigma M\otimes\Sigma N}(\psi^+|_M) = 0, \qquad \nabla^{\Sigma M\otimes\Sigma N}(\psi^-|_M) = 0$$

for any parallel spinor field $\psi^+ \in \Gamma(\Sigma^+ Q)$ and $\psi^- \in \Gamma(\Sigma^- Q)$.

(4)⇒(3): Let $\psi^+ \in \Gamma(\Sigma^+ Q)$ and $\psi^- \in \Gamma(\Sigma^- Q)$ be parallel spinor fields on Q such that $|\psi^+| \equiv 1$ and $|\psi^-| \equiv 1$. By Eq. (5), we have

$$0 = \nabla^{\Sigma Q}_X(\psi^+|_M) - \nabla^{\Sigma M\otimes\Sigma N}_X(\psi^+|_M) = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\langle II(X, X_i), Y_j\rangle\gamma_Q(X_i\cdot Y_j)\psi^+|_M, \tag{10}$$

and

$$0 = \nabla^{\Sigma Q}_X(\psi^-|_M) - \nabla^{\Sigma M\otimes\Sigma N}_X(\psi^-|_M) = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}\langle II(X, X_i), Y_j\rangle\gamma_Q(X_i\cdot Y_j)\psi^-|_M \tag{11}$$

for any $X \in \Gamma(TM)$. To obtain the condition (3), it suffices to show that

$$\langle II(X, X_i), Y_j\rangle(p) = 0, \quad i, j = 1, 2$$

for each point $p \in M$. Fix $p \in M$ and put $\varphi := \psi^+|_M + \psi^-|_M$. Of course, $\langle\psi^+|_M, \psi^+|_M\rangle = \langle\psi^-|_M, \psi^-|_M\rangle = 1$ and $\langle\psi^+|_M, \psi^-|_M\rangle = 0$. From Eqs. (10) and (11) it follows that

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\langle II(X, X_i), Y_j\rangle\gamma_Q(X_i\cdot Y_j)\varphi = 0.$$

By taking Hermitian inner product with $\gamma_Q(X_1\cdot Y_1)\varphi$, we have

$$\begin{aligned} &2\langle II(X, X_1), Y_1\rangle + \langle II(X, X_1), Y_2\rangle\langle\gamma_Q(Y_2)\varphi, \gamma_Q(Y_1)\varphi\rangle \\ &\quad + \langle II(X, X_2), Y_1\rangle\langle\gamma_Q(X_2)\varphi, \gamma_Q(X_1)\varphi\rangle \\ &\quad + \langle II(X, X_2), Y_2\rangle\langle\gamma_Q(X_2\cdot Y_2)\varphi, \gamma_Q(X_1\cdot Y_1)\varphi\rangle = 0. \end{aligned} \tag{12}$$

Since

$$\begin{aligned} &\langle\gamma_Q(X_2\cdot Y_2)\varphi, \gamma_Q(X_1\cdot Y_1)\varphi\rangle \\ &\quad = \langle\varphi, \gamma_Q(Y_2\cdot X_2\cdot X_1\cdot Y_1)\varphi\rangle = \langle\varphi, \gamma_Q(X_1\cdot X_2\cdot Y_1\cdot Y_2)\varphi\rangle \\ &\quad = \langle\varphi, -\gamma_Q(\omega_{\mathbf{C}})\varphi\rangle = \langle\psi^+|_M + \psi^-|_M, -\psi^+|_M + \psi^-|_M\rangle = 0, \end{aligned}$$

the real part of Eq. (12) is

$$2\langle II(X, X_1), Y_1\rangle = 0.$$

Similarly, we obtain

$$\langle II(X, X_1), Y_2\rangle(p) = 0, \qquad \langle II(X, X_2), Y_1\rangle(p) = 0, \qquad \langle II(X, X_2), Y_2\rangle(p) = 0.$$

Hence we have the condition (3). □

The equivalence of (1) and (3) is a well-known result, but the above theorem gives an alternative proof of the fact.

Finally, we give a result on four-dimensional hyperkähler manifolds for that problem. In contrast to a four-dimensional flat torus, a four-dimensional *nonflat* hyperkähler manifold has a nonzero positive parallel spinor field, but has no negative one (see [2, Chapter 6]).

Theorem 4.5. *Let Q^4 be a four-dimensional hyperkähler manifold. Let M^2 be a closed Riemann surface immersed in Q^4. Then the following conditions are equivalent*:

1. *M^2 is a minimal surface in Q^4 such that $\chi(M) + \chi(N) = 0$.*
2. *M^2 is a holomorphic curve with respect to one of the complex structures on Q^4 compatible with the metric.*
3. *M^2 satisfies the condition* $(*)$.
4. $\nabla^{\Sigma M\otimes\Sigma N}(\psi^+|_M) = 0$ *for any parallel spinor field $\psi^+ \in \Gamma(\Sigma^+ Q)$.*

Proof. (1)⇒(4): Trivial by Proposition 4.2(1).

(4)⇒(3): Trivial.

(3)⇒(1): By Proposition 3.1, M^2 is a minimal surface in Q^4. Hence Proposition 4.2(1) yields the condition (1).

Therefore, it suffices to show the equivalence of (1) and (2).

(1)⇒(2): Webster proved that under our situation,

$$-P - Q = \chi(M) + \chi(N), \tag{13}$$

where P is the number of complex tangent points and Q is the number of anti-complex tangent points of M^2 in Q^4 (see [9]). Since $\chi(M) + \chi(N) = 0$, we have $P = Q = 0$. This means that M^2 is a totally real surface in Q^4. By Wolfson's theorem (see [9, Theorem 2.2] and [7, Section 2]), we obtain the condition (2).

(2)⇒(1): Q^4 has a hyperkähler structure I, J, K. If M^2 is a holomorphic curve with respect to the complex structure I, then M^2 is a Lagrangian surface in Q^4 with respect to the Kähler form defined by the complex structure J. Hence M^2 is a totally real surface in Q^4. By Eq. (13), we have $\chi(M) + \chi(N) = 0$. □

We remark that the equivalence of (1) and (2) in the above theorem is already known by Micallef and Wolfson [7, Section 1].

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